Instructions: The bold letters $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ represent the unit vectors in the x, y, and z directions, respectively, in 3-dimensional space. The floor function, denoted as $\lfloor x \rfloor$, represents the greatest integer less than or equal to x. NOTA means "None of the Above". Good luck and have fun!

1) Compute $\begin{vmatrix} 3 & 2 \\ 7 & 68 \end{vmatrix}$.

A. -218 B. -190 C. 190 D. 218 E. NOTA

2) Alice is a hunter that lives on the plane $3x + 2y + 2z = 16$. She sees prey at the location $(x, y, z) = (3,5,7)$.

If (a, b, c) is the closest point she can reach to her prey, compute $a + b + c$.

A. 5 B. 6

C. 7 D. 8 E. NOTA

For questions 3 to 4, consider the matrix $M = \begin{bmatrix} 3 & -2 \\ 5 & 14 \end{bmatrix}$.

3) Which of the following is an eigenvector of M?

A. $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ E. NOTA

4) What is the sum of the elements of the reduced row echelon form of M?

A. $\frac{4}{3}$ B. 2 C. $\frac{24}{5}$ D. $\frac{53}{3}$ E. NOTA

5) Find the value in the second row and first column in the cofactor matrix of $\begin{bmatrix} -2 & -4 & -2 \\ 4 & -7 & -1 \\ -4 & -8 & 6 \end{bmatrix}$.

A. -40 B. -8 C. 8 D. 40 E. NOTA

6) Exactly which of the following statements are true regarding elementary matrices?

I) An elementary matrix must be triangular

II) An elementary matrix must have at least 1 zero element

III) Multiplying a matrix by an elementary matrix on the left is equivalent to performing a row operation on that matrix.

A. I B. I and II C. I and III D. II and III E. NOTA

7) Which of the following matrices is idempotent?

A. $\begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 5 & 14 \\ 1 & 3 & 8 \\ -1 & -2 & -6 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 5 & -5 \\ 15 & 3 & 5 \\ 10 & -2 & -4 \end{bmatrix}$ D. $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ E. NOTA

8) Compute the value of $\left\|\langle\sqrt{17}, 2\sqrt{17}, 3\sqrt{17}\rangle \times \langle 1-\sqrt{17}, 2-\sqrt{17}, 3-\sqrt{17}\rangle\right\|$.

A. 17 B. 102 C. $17\sqrt{6}$ D. 189 E. NOTA

For question 9 the following information may be helpful:

A right-handed system is one in which $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$. This is where the “right-hand rule” in physics originates from and is the default convention in linear algebra. For a left-handed system, dot products work the same, but the cross product is the negation of what it would have been in a right-handed system (e.g., $\boldsymbol{i} \times \boldsymbol{j} = -\boldsymbol{k}$).

9) How many of the following statements hold in a left-handed system?

I) Commutativity over cross products

II) Associativity over cross products

III) Cross products distribute over addition

IV) Cross products scale over constant factors

A. 1 B. 2 C. 3 D. 4 E. NOTA

10) An ant is traveling along the vertices of a cube. Every minute, it moves to an adjacent vertex at random. Find the probability of the ant being at the starting vertex after 4 minutes.

A. $\frac{1}{9}$ B. $\frac{5}{27}$ C. $\frac{7}{27}$ D. $\frac{1}{3}$ E. NOTA

11) Which of the following are characteristics of linear transformations?

I) Linear transformations can be uniquely defined by how they operate on the unit vectors.

II) Linear transformations can only be applied to lines (vectors).

III) Linear transformations are defined up to 3 dimensions.

A) I B) I and II C) I and III D) II and III E) NOTA

12) Compute the value of $\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}^{100}$

A. $\begin{bmatrix} -4-3\cdot 2^{100} & -4+2^{102} \\ -3-3\cdot 2^{100} & -3+2^{102} \end{bmatrix}$ B. $\begin{bmatrix} -4-3\cdot 2^{100} & 4+2^{102} \\ -3-3\cdot 2^{100} & 3+2^{102} \end{bmatrix}$

C. $\begin{bmatrix} 4-3\cdot 2^{100} & -4+2^{102} \\ 3-3\cdot 2^{100} & -3+2^{102} \end{bmatrix}$ D. $\begin{bmatrix} 4-3\cdot 2^{100} & 4+2^{102} \\ 3-3\cdot 2^{100} & 3+2^{102} \end{bmatrix}$ E. NOTA

13) What is the minimum distance between the lines $x+2=2y+3=3z+4$ and $3x+5=2y+4=z+2$.

A) 0 B) $\frac{\sqrt{82}}{82}$ C) $\frac{3\sqrt{82}}{82}$ D) $\frac{5\sqrt{82}}{82}$ E) NOTA

14) Suppose we have a 3×3 binary matrix (each element is either 0 or 1). What is the minimum number of nonzero elements that would guarantee that this matrix is not nilpotent?

A) 3 B) 4 C) 5 D) 6 E) NOTA

15) Find the real value of k such that the following matrix is singular: $\begin{bmatrix} 3+2k & 2k \\ k^2 & 6 \end{bmatrix}$.

A) $3-\sqrt{3}$ B) 3 C) $3+\sqrt{3}$ D) $3\sqrt{3}$ E) NOTA

16) A system with 4 states (State 1, State 2, State 3, and State 4) has a probability transition matrix of $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 & \frac{2}{5} \\ \frac{3}{7} & \frac{2}{7} & 0 & \frac{2}{7} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$. The element in row i and column j is the probability of the transition from State i to State j. If the system begins in State 1, after a long time, what is the probability of the system still being in State 1?

A) 0 B) $\frac{3}{7}$ C) $\frac{13}{35}$ D) $\frac{7}{60}$ E) NOTA

17) The Coriolis force is an apparent force when an object has a velocity in a rotating reference frame. The formula for the Coriolis force is $F_c = -2m\Omega \times v$ where m is the mass of an object, Ω is the angular velocity of the reference frame, and v is the velocity of the object in the rotating reference frame. Given that the angular velocity of the Earth is $7.3 \cdot 10^{-5}\ s^{-1}$, what is the magnitude of the Coriolis force, in Newtons, on a cloud at the equator with mass $4000\ kg$, and velocity $30\frac{m}{s}$ directed 30 degrees north of east?

A) $\frac{219\sqrt{3}}{25}$ B) $\frac{438\sqrt{3}}{25}$ C) $\frac{219}{25}$ D) $\frac{438}{25}$ E) NOTA

18) Compute $\langle 1,2,3\rangle \cdot \langle 4,5,6\rangle$.

A) 8 B) 16 C) 32 D) 64 E) NOTA

19) The eigenvalues of the matrix $\begin{bmatrix} a & 1 \\ 1 & c \end{bmatrix}$ are the minimum and maximum values of $f(x) = |\sin x| + |\cos x|$. Compute $a^2 + c^2$.

A) $\sqrt{2} - 1$ B) 1 C) $\sqrt{2} + 1$ D) 3 E) NOTA

For problems 20 to 22 the following information may be helpful:

LU Decomposition is a factorization of a matrix into upper and lower triangular matrices, or matrices that have all zeroes below and above the main diagonal, respectively. For example, $\begin{bmatrix} 4 & 5 \\ 8 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$. Note that the matrices in the product are **L**ower and **U**pper triangular, respectively.

20) Consider the matrix $\begin{bmatrix} 48 & 36 \\ 12 & 23 \end{bmatrix}$. Given that a LU Decomposition of this matrix is $\begin{bmatrix} 48 & 36 \\ 12 & 23 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 4 & c \\ 0 & d \end{bmatrix}$, what is $abcd$?

A) 48 B) 72 C) 105 D) 126 E) NOTA

21) Exactly which of $M = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ can be decomposed with LU Decomposition?

A) Neither B) M C) N D) Both E) NOTA

22) How many of the following matrices have an LU decomposition with lower and upper matrices that have main diagonal elements all integers greater than 1?

I) $\begin{bmatrix} 4 & 3 \\ 7 & 3 \end{bmatrix}$ II) $\begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ III) $\begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$ IV) $\begin{bmatrix} 3 & 2 \\ 5 & 9 \end{bmatrix}$

A) 1 B) 2 C) 3 D) 4 E) NOTA

23) Two planes intersect along the line $L = \langle a, b, c \rangle + t\langle 2, -11, 8 \rangle$, for some constants a, b, and c. One of the planes in question has equation $5x + 6y + 7z = 8$. Which of the following could be the equation of the other plane?

A. $x + 2y + 3z = 5$
B. $-x + 2y + 3z = 6$
C. $x + 2y - 3z = 7$
D. $x - 2y + 3z = 8$
E. NOTA

24) The angle between the vectors $\vec{m} = \langle 1, 7\rangle$ and $\vec{n} = \langle 7,1\rangle$ is bisected by the vector $\vec{o}$. Compute the cosine of the angle between $\vec{n}$ and $\vec{o}$.

A) $\frac{1}{2}$ B) $\frac{3}{5}$ C) $\frac{4}{5}$ D) $\frac{\sqrt{2}}{2}$ E) NOTA

25) Consider the following 4 matrices. Matrix A is a 2×3 matrix. Matrix B is a 1×4 matrix. Matrix C is a 3×4 matrix. Matrix D is a 4×2 matrix. How many valid multiplication pairs are possible, accounting for order?

A) 4 B) 5 C) 6 D) 7 E) NOTA

26) Consider the line $y = x\sqrt{3}$. What matrix can left-multiplied to transform this to the line $y = -x$?

A. $\begin{bmatrix} \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \end{bmatrix}$ B. $\begin{bmatrix} \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \end{bmatrix}$

C. $\begin{bmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \\ \frac{-\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \end{bmatrix}$ D. $\begin{bmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{-\sqrt{6}-\sqrt{2}}{4} \\ \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \end{bmatrix}$ E. NOTA

27) Anthony applied shoelace to 11 different quadrilaterals and got the raw values $20, -8, 2, -19, 15, -9, -12, -2, -20, 15, -11$. Note that these are before taking the absolute value and dividing by 2. All of the quadrilaterals contained the point $(3, 7)$ in the interior. How many of the quadrilaterals had vertices arranged so that $(3, 7)$ was on the left-hand side of the path at all times? For example, going from the origin to $(2, 4)$ would have $(3, 7)$ on the left side because the line passing through O and $(2, 4)$ is to the right of $(3, 7)$.

A) 4 B) 5 C) 6 D) 7 E) NOTA

28) Consider the matrix $M = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 3 & 7 \\ 1 & 0 & 4 \end{bmatrix}$. Compute the sum of the complex eigenvectors taken 2 at a time.

A) -22 B) -11 C) 11 D) 22 E) NOTA

29) Consider a unit vector $\vec{v}$ and another vector $\vec{w}$ with a magnitude of 2. If $\|\vec{v} + \vec{w}\| = 2\sqrt{2}$, compute $\|\vec{v} - \vec{w}\|$.

A) $\frac{1}{2}$ B) $\frac{\sqrt{2}}{2}$ C) 1 D) $\sqrt{2}$ E) NOTA

30) Congratulations! You made it to the last question. What is the trace of the 10 by 10 identity matrix?

A) 0 B) 1 C) 10 D) 100 E) NOTA