1. A
2. D
3. B
4. C
5. E
6. C
7. D
8. B
9. C
10. C
11. C
12. B
13. E
14. A
15. D
16. B
17. C
18. B
19. E
20. D
21. D
22. A
23. B
24. C
25. D
26. D
27. A
28. B
29. C
30. D

1. A $20^2 - 19^2 = (20-19)(20+19) = 39$.
2. D Call the matrix M, and let n be a natural number such that $M^n = 0$ (the 0 matrix), which must be possible for M to be nilpotent. If M is idempotent, then $M^2 = M, M^3 = M^2 * M = M * M = M, M^4 = M^3 * M = M * M = M, \dots, M^n = M^{n-1} * M = M * M = M$, where n is the same value that satisfies $M^n = 0$. Now, we have $M^n = 0$ and $M^n = M$, so $M = 0$. This means M is the 0 matrix (all the elements of M are 0). The other choices are necessary but not sufficient.
3. B $\langle 5,4,3\rangle \cdot \langle x,y,z\rangle = \langle 5x, 3y, z\rangle \cdot \langle 1, \frac{4}{3}, 3\rangle \le \|\langle 5x,3y,z\rangle\| * \left\|\langle 1,\frac{4}{3},3\rangle\right\| = \sqrt{35} * \sqrt{1+\frac{16}{9}+9} = \sqrt{35} * \frac{\sqrt{9+16+81}}{3} = \frac{\sqrt{35*106}}{3} = \frac{\sqrt{3710}}{3}$
4. C We can find the characteristic polynomial by taking the determinant of $\begin{bmatrix} -7-\lambda & 0 & 4 \\ -12 & 1-\lambda & 6 \\ -20 & 0 & 11-\lambda \end{bmatrix}$, which is $(1-\lambda)\big((-7-\lambda)(11-\lambda) - (-20)(4)\big) = (1-\lambda)(\lambda^2 - 4\lambda - 77 + 80) = (1-\lambda)(\lambda-3)(\lambda-1)$. Thus the eigenvalues are $1, 1, 3$.
5. E All of the choices are eigenvectors. This can be verified by multiplying matrix M to the left of each choice. A, B, and D are associated with the eigenvalue of 1, while C is associated with the eigenvalue of 3.
6. C From number 4, the eigenvalues of M are 1 (with algebraic multiplicity 2) and 3 (with algebraic multiplicity 1). 2 is not an eigenvalue so it has a geometric multiplicity of 0. The question boils down to finding the geometric multiplicity of the eigenvalue 1. The eigenvectors associated with this eigenvalue satisfy $M\vec{v} = \vec{v} \rightarrow$ $\begin{bmatrix} -7 & 0 & 4 \\ -12 & 1 & 6 \\ -20 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. This means $\begin{matrix} -7x_1 + 4x_3 = x_1 \\ -12x_1 + x_2 + 6x_3 = x_2 \\ -20x_1 + 11x_3 = x_3 \end{matrix}$
   All 3 equations simplify to $x_3 = 2x_1$, and x_2 can be any value.
   Since there are 2 free variables, the geometric multiplicity is 2.
7. D The trace is the sum of the elements along the main diagonal. Along the main diagonal, $i = j$, so we want $\sum_{i=1}^{100}(2i) = 2\sum_{i=1}^{100} i = 2 * \frac{(100*101)}{2} = 10100$
8. B It is possible, but not easiest, to solve for x using Cramer's rule. Instead, note that the coefficient of y and the final sum are always in a ratio of $3:1$. This means $y = \frac{1}{3}, w = x = z = 0$ satisfies the equation. Since there are an equal number of variables and equations, this is the unique solution. $x = 0$
9. C The area is half of the product of the base and height, which is half the magnitude of the cross product of the vectors representing adjacent edges. Two of these vectors are $(2,3,4) - (1,1,1) = \langle 1,2,3\rangle$, and $(-4,5,7) - (1,1,1) = (-5,4,6)$. $\begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ -5 & 4 & 6 \end{bmatrix} = i * (2*6 - 4*3) - j(1*6 + 3*5) + k(1*4 + 2*5) = i * (12-12) - j(6+15) + k(4+10) = \langle 0,-21,14\rangle. |\langle 0,-21,14\rangle| = 7|\langle 0,-3,2\rangle| = 7\sqrt{3^2+2^2} = 7\sqrt{13}$
   So the area is $7\sqrt{13}/2$

10. C $(2M) = 2\,tr(M)$ because each element along the main diagonal is multiplied by 2, and these are all added together. $\det(2M) = \det(M) * 2^n$, so the ratio $\frac{\det(2M)}{tr(2M)} = \frac{\det(M)}{tr(M)} * \frac{2^n}{2} = 1 * 2^{n-1} = 2^{n-1}$

11. C I is true. II is false; a 2 x 1 matrix could have a reduced row echelon form of $\begin{bmatrix}1\\0\end{bmatrix}$. III is false. While the description of the columns does match any column with a leading 1 in it, there is nothing preventing trailing columns to follow the same pattern.

12. B Generally, this system will have exactly one solution. However, if $\frac{a}{b} = \frac{c}{d}$, the systems become dependent. In this case, either the system has no solutions (if the system is inconsistent) or an infinite number of solutions (if the system is consistent). Since the general case results in 1 solution, k must be chosen so the systems are consistent if $\frac{a}{b} = \frac{c}{d}$. This results if one equation is a multiple of the other: $(ax + by) * n = (cx + dy), nk = 2$. From the first equation, $n = \frac{c}{a} = \frac{d}{b}$. From the second, $k = \frac{2}{n} = \frac{2a}{c} = \frac{2b}{d}$.

13. E In order for the matrix to be singular, the determinant must be 0: $\begin{vmatrix} x & 6 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 8 \end{vmatrix} = 0 \rightarrow$ $x(2*8 - 7*3) - 6(4*8 - 9*3) + 5*(4*7 - 9*2) = x*(16 - 21) - 6(32 - 27) + 5*(28 - 18) = -5x - 6*5 + 5*10 = -5x - 30 + 50 = -5x + 20 = 0 \rightarrow x = 4$

14. A The reasons these matrices are called orthogonal is because each pair of unique rows is orthogonal (i.e., they have a dot product of 0). So II is possible. Since the rows are chosen independently, it is possible the same row is chosen twice, which would result in a dot product of 1. So III is also possible. II and III are possible

15. D The roots of the characteristic polynomial are the eigenvalues, so the sum of the roots is the sum of the eigenvalues, which is just the trace. $4 + 6 - 7 = 3$

16. B Writing out the matrix using the rules described is possible but unnecessary. Instead, the combinatorial rules which apply in Pascal's triangle can be extended to this situation. Specifically, if $A = P_n$, $A_{i,j} = {}_{i+j-2}C_{i-1} = {}_{i+j-2}C_{j-1}$. So, we want $A_{5,5} = {}_8C_4 = \frac{8*7*6*5}{4*3*2*1} = \frac{40}{4} * 7 * \frac{6}{6} = 10 * 7 = 70$

17. C Let $A = P_n$. Since $A_{i,j} = {}_{i+j-2}C_{i-1}$,the sum of the elements in the a^{th} column of A $= {}_{a-1}C_{a-1} + {}_aC_{a-1} + {}_{a+1}C_{a-1} \ldots + {}_{a+n-2}C_{a-1} = {}_{a+n-1}C_a$, where the last simplification was using the hockey stick identity. Since $n = 9$, and $a = 3$, the value requested $= {}_{3+9-1}C_3 = {}_{11}C_3 = \frac{11*10*9}{6} = 11 * 15 = 165$.

18. B Since squaring positive numbers doesn't change the relationships between them, we can consider the square of the magnitude for each choice. For A, this is $8^2 + 8^2 + 8^2 = 64 * 3 = 192$. For B, $11^2 + 5^2 + 7^2 = 121 + 25 + 49 = 195$. For C, $12^2 + 3^2 + 6^2 = 144 + 9 + 36 = 185$. For D, $3^2 + 8^2 + 10^2 = 9 + 64 + 100 = 173$. For E, $13^2 + 4^2 + 3^2 = 169 + 16 + 9 = 194$. Thus 195 is the greatest.

19. E The determinant of the matrix is 0 because the second row is the sum of the first and third rows. So, the matrix is singular (i.e., it has no inverse). Answer choice A

provides the correct adjugate matrix, but this then needs to be divided by the determinant, which is division by 0.

20. D Converting the lines to parametric, they become $x = t, y = 2t + 1, z = 6t - 25$ and $x = 4s, y = s + 8, z = -s$. The vectors in the direction of each line are $\langle 1,2,6 \rangle$ and $\langle 4,1,-1 \rangle$, respectively. $\langle 1,2,6 \rangle \cdot \langle 4,1,-1 \rangle = 4 + 2 - 6 = 0$, so, the vectors are orthogonal. Now, whether the lines intersect will determine if they are skew or are perpendicular. Setting the x-values equal, $t = 4s$. Setting the y-values equal, $2t + 1 = 2 * 4s + 1 = s + 8 \rightarrow 8s + 1 = s + 8 \rightarrow 7s = 7 \rightarrow s = 1$. The z values are $6t - 25 = 6 * (4s) - 25 = 6 * 4 - 25 = -1$, and $-s = -1$, so this works. The lines do intersect, and are at right angles to each other, so they are perpendicular.

21. D The scalar projection is the magnitude of the vector projection, so it is $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\|\vec{v}\| = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \frac{-8-6+20}{\sqrt{16+4+25}} = \frac{6}{\sqrt{45}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

22. A I is true; the first row or column can be chosen (depending on whether the matrix is upper triangular or lower triangular) for minor expansion since it has only 1 element and this can be done repeatedly. Only the elements on the main diagonal end up being used. II is true; any element that only has elements along the main diagonal (such as the identity matrix) is both upper triangular and lower triangular. III is true; specifically, the transpose of an upper triangular matrix is a lower triangular matrix, and the transpose of a lower triangular matrix is an upper triangular matrix. IV is true; inverse of an upper/lower triangular matrix is also an upper/lower triangular matrix.

23. B Using the definition of an inverse ($M * M^{-1} = I$) requires the least work in this scenario because all that is necessary is solving a single linear equation. $M * (M^{-1} * \det(M)) = (M * M^{-1}) * \det(M) = I * \det(M)$, which is a matrix with elements equals to $\det(M)$ along the main diagonal and 0 everywhere else. Now, it is possible to equate specific elements of these matrices: $(I * \det(M))_{1,3} = 0$. $\left(M * (M^{-1} * \det(M))\right)_{1,3} = 1 * x - 3 * -15 + 4 * -96 + 2 * 109 = x + 45 - 384 + 218 = x - 121 = (I * \det(M))_{1,3} = 0 \rightarrow x = 121$

24. C The determinant of the matrix is $3a - 2b$, so it must be the case that $3a - 2b \neq 0 \rightarrow 3a \neq 2b$. This is true only for choice C.

25. D In order to find the minimum distance, one possible approach is to find a plane containing one of the lines that is parallel to the other line. This requires a plane with a normal vector that is perpendicular to both lines, which can be computed by taking the cross product of the lines. Then, a point on the other line (not on the plane) can be chosen and it is a point to plane problem. In line with the above approach, the vectors in the directions of the lines are $\langle 2,3,-1 \rangle$ and $\langle 1,-2,1 \rangle$. The cross product is $\begin{bmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} = i(3-2) - j(2+1) + k(-4-3) = \langle 1,-3,-7 \rangle$. The equation of the plane is $x - 3y - 7z = C$. If the first line is chosen to be on the plane, one point on the first line is $(0,-1,3)$ at $t = 0$. So, $C = 0 - 3 * -1 - 7 * 3 - 0 + 3 - 21 = -18$. A point on the second line is $(-7,4,1)$ at $s = 0$ and the distance from this point to the plane is $\frac{|1*-7-3*4-7*1+18|}{\sqrt{1+9+49}} = \frac{|18-7-12-7|}{\sqrt{59}} = \frac{8}{\sqrt{59}} = \frac{8\sqrt{59}}{59}$

26. D $\begin{vmatrix} 1 & 4 & 5 \\ 2 & 0 & 6 \\ -4 & 3 & 7 \end{vmatrix} = 1*(-18) - 4*(14+24) + 5*(6) = -18 - 4*38 + 30 =$ $-18 + 30 - 152 = -140$. Since we want the volume of the parallelepiped, we want $|-140| = 140$

27. A Since 21000 games is a very large number of games, the number of games A would be expected to win would be very close to the steady state probability of A winning multiplied by 21000. $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$ represents the transition matrix for this situation, where the rows represent the next result (row 1 = win, row 2 = draw, row 3 = loss), the column represent the previous result (column 1 = win, column 2 = draw, column 3 = loss), and element i, j represents the probability of outcome j given the previous outcome i. Note that everything is being considered from the perspective of A. The steady state probability of winning satisfies $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w \\ \frac{1}{5} \\ \frac{4}{5} - w \end{bmatrix} = \begin{bmatrix} w \\ \frac{1}{5} \\ \frac{4}{5} - w \end{bmatrix}$, so, $\begin{bmatrix} \frac{-2}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} w \\ \frac{1}{5} \\ \frac{4}{5} - w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $-2w + \frac{2}{5} + \frac{3}{2}\left(\frac{4}{5} - w\right) = 0 \rightarrow -20w + 4 +$ $3(4 - 5w) = -20w - 15w + 4 + 12 = 0 \rightarrow 35w = 16 \rightarrow w = \frac{16}{35}$. $21000w =$ $21000 * \frac{16}{35} = 200 * 105 * \frac{16}{35} = 200 * 3 * 16 = 9600$ wins.

28. B $$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{0 + 56 + 576}{\sqrt{7^2 + 24^2}\sqrt{6^2 + 8^2 + 24^2}} = \frac{632}{25 * \sqrt{10^2 + 24^2}} = \frac{632}{25 * 26} = \frac{316}{325}$$

29. C The normal vector to the plane is $\langle 1,1,-1 \rangle$. The projection of $\langle 1,4,-3 \rangle$ onto $\langle 1,1,-1 \rangle$ is $\frac{\langle 1,1,-1 \rangle \cdot \langle 1,4,-3 \rangle}{\|\langle 1,1,-1 \rangle\|^2} \langle 1,1,-1 \rangle = \frac{1+4+3}{\sqrt{3}^2} \langle 1,1,-1 \rangle = \frac{8}{3} \langle 1,1,-1 \rangle$. $\langle 1,4,-3 \rangle -$ $\langle \frac{8}{3}, \frac{8}{3}, -\frac{8}{3} \rangle = \langle -\frac{5}{3}, \frac{4}{3}, -\frac{1}{3} \rangle$

30. D I is true; the trace of a matrix is the sum of the elements along the main diagonal. II is true; this is just the determinant produced by minor expansion of M with $-b(di - fg)$ converted to $b(fg - di)$. III is true. I, II, III are all true.